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Normality of meromorphic functions with multiple zeros and shared values

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Abstract

In this paper, we study the normality of a family of meromorphic functions and general criteria for normality of families of meromorphic functions with multiple zeros concerning shared values are obtained.

Keywords: Meromorphic function; Differential polynomial; Shared value; Normality

1. Introduction

Let $f(z)$ be a meromorphic function on a domain D in the complex plane, m, q, k be positive integers, $a_i(z)$ $(i = 1, 2, \dots, q-1)$, $b_j(z)$ $(j = 1, 2, \dots, m)$ be analytic functions in D, $n_0, n_1, \dots, n_k$ be nonnegative integers. Set

$$P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \cdots + a_1(z)\omega,$$

$$M\left(f, f', \dots, f^{(k)}\right) = f^{n_0}(f')^{n_1} \dots \left(f^{(k)}\right)^{n_k},$$

$$\gamma_M^* = n_0 + n_1 + \cdots + n_{k-1},$$

$$\gamma_M = n_0 + n_1 + \cdots + n_k,$$

$$\Gamma_M = n_0 + 2n_1 + 3n_3 + \cdots + (k+1)n_k.$$

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$M(f, f', \dots, f^{(k)})$ is called the differential monomial in f. Further, let $M_1(f, f', \dots, f^{(k)})$, $M_2(f, f', \dots, f^{(k)})$, $\dots$, $M_m(f, f', \dots, f^{(k)})$ be differential monomials in f. We call

$$H\big(f, f', \dots, f^{(k)}\big) = b_1(z)M_1\big(f, f', \dots, f^{(k)}\big) + \dots + b_n(z)M_n\big(f, f', \dots, f^{(k)}\big)$$

the differential polynomial in f and we define

$$\gamma_H^* = \min\{\gamma_{M_1}^*, \gamma_{M_2}^*, \dots, \gamma_{M_m}^*\}.$$

Let f and g be meromorphic functions on a domain D, and let a and b be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = b$. If $f(z) = a \Rightarrow g(a) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = b$. If $f(z) = a \Leftrightarrow g(z) = a$, then we say that f and g share a in D.

Schwick first showed a connection between normality criteria and shared values. He proved

Theorem A [1]. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and let a_1, a_2, and a_3 be distinct complex numbers. If f and f' share a_1, a_2, and a_3 for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In 2000, Pang and Zalcman proved

Theorem B [2]. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and let a and b be distinct complex numbers. If, for every $f \in \mathcal{F}$, f and f' share a and b then $\mathcal{F}$ is normal in D.*

Naturally, we ask what can be stated if f' is replaced by a differential polynomial in f in Theorem B. In this paper, we prove

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, k, m, q be positive integers, $P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \dots + a_1(z)\omega$ be a polynomial, and let $H(f, f', \dots, f^{(k)})$ be a differential polynomial as above stated which satisfies $\gamma_H^* > 0$, $a(z), b(z) \neq 0$, $c(z) \neq 0$ be some analytic functions in D. If, for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least k, and*

$$f(z) = 0 \quad \Leftrightarrow \quad P\big(f^{(k)}\big) + H\big(f, f', \dots, f^{(k)}\big) = a(z) \quad \text{and}$$
$$P\big(f^{(k)}\big) + H\big(f, f', \dots, f^{(k)}\big) = b(z) \quad \Rightarrow \quad f(z) = c(z),$$

then $\mathcal{F}$ is normal in D for $k \geqslant 2$, and for $k = 1$ so long as $a(z) \neq (m+1)b(z)$ $(m = 1, 2, \dots)$.

If $a(z) \equiv b(z)$, we can get the following corollary.

Corollary. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, k, n, q be positive integers, $P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \dots + a_1(z)\omega$ be a polynomial, and let*

$H(f, f', \dots, f^{(k)})$ be a differential polynomial which satisfies $\gamma_H^* > 0$, $b(z)$ analytic in D and $b(z) \neq 0$. If, for every $f \in \mathcal{F}$,

$$f(z) \neq 0 \quad \text{and} \quad P\big(f^{(k)}\big) + H\big(f, f', \dots, f^{(k)}\big) \neq b(z),$$

then $\mathcal{F}$ is normal in D.

In 2001, Chen and Fang proved

Theorem C [3]. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let $k \geqslant 2$ be a positive integer, and let a, b, c be complex numbers such that $a \neq b$. If, for each $f \in \mathcal{F}$, f and $f^{(k)}$ share a and b in D, and the zeros of $f(z) - c$ are of multiplicity $\geqslant k+1$, then $\mathcal{F}$ is normal in D.*

In this paper, we improve the above result and get the following theorem.

Theorem 2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer, and let a, b, c be complex numbers such that $a \neq b$. If, for each $f \in \mathcal{F}$, f and $f^{(k)}$ share a and b in D, and the zeros of $f(z) - c$ are of multiplicity $\geqslant k$, then $\mathcal{F}$ is normal in D.*

2. Some lemmas

Lemma 1 [4]. *Let k be a positive integer and let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ all of whose zeros have multiplicity at least k, and suppose that there exists $A \geqslant 1$ such that $|f^{(k)}(z)| \leqslant A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal at origin, there exist, for each $0 \leqslant \alpha \leqslant k$,*

(a) *a number r, $0 < r < 1$,*
(b) *points z_n, $|z_n| < r < 1$, $z_n \to 0$,*
(c) *functions $f_n \in \mathcal{F}$, and*
(d) *positive numbers $\rho_n \to 0^+$,*

such that

$$g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C such that $g^{\#}(\xi) \leqslant g^{\#}(0) = kA + 1$. Moreover, g is of order at most two.

Remark. In Lemma 1, if taken $\alpha = 0$, then $g(\xi)$ is a meromorphic function on C which satisfies $g^{(k)}(\xi) \not\equiv 0$ (see [5]).

Lemma 2 [6]. *Let g be a meromorphic function with finite order. If g has only finitely many critical values, then it has only finitely many asymptotic values.*

Lemma 3 [7]. *Let g be a transcendental meromorphic function such that $g(0) \neq \infty$ and the set of finite critical and asymptotic values of g is bounded. Then there exists $R > 0$ such that*

$$|g'(z)| \geqslant \frac{|g(z)|}{2\pi|z|} \log \frac{|g(z)|}{R}$$

for all $z \in C \backslash \{0\}$ which are not poles of g.

Lemma 4 [8]. *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + q(z)/p(z)$ where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$, $q(z)$ and $p(z)$ are two coprime polynomials, neither of which vanishes identically, with $\deg q(z) < \deg p(z)$; and let k be a positive integer. If $f^{(k)}(z) \neq 1$, then*

$$f(z) = \frac{z^k}{k!} + \cdots + a_0 + \frac{A}{(z-d)^m}.$$

Here, $A \neq 0$ be a constant and m be a positive integer.

Lemma 5 [10]. *Suppose that $f(z)$ is meromorphic and transcendental in the plane. Then as $r \to \infty$*

$$T(r, f) \leqslant \left(2 + \frac{1}{l}\right) N\left(r, \frac{1}{f}\right) + \left(2 + \frac{2}{l}\right) \overline{N}\left(r, \frac{1}{f^{(l)} - 1}\right) + S(r, f).$$

Lemma 6. *Let f be a nonconstant meromorphic function of finite order, a and b be distinct complex numbers and $b \neq 0$. If $f(z) = 0 \Leftrightarrow f' = a$ and $f'(z) \neq b$ in C, then*

$$f(z) = b(z-d) + \frac{A}{m(z-d)^m} \quad \text{and} \quad a = (m+1)b$$

for some $d \in C$ and some positive integer m.

Proof. Suppose that $f(z)$ is a transcendental meromorphic function, we can deduce $f(z)$ has infinitely many zeros $z_1, z_2, \ldots, z_n, \ldots$ by the assumption and Lemma 5. Define $g(z) = f(z) - bz$; then $g'(z) = f' - b$. It is easy to see that $g(z)$ is a meromorphic function with finite order and $g'(z) \neq 0$. Hence, by Lemmas 2 and 3, there exists $R > 0$ such that

$$\frac{|z_n g'(z_n)|}{|g(z_n)|} \geqslant \frac{1}{2\pi} \log \frac{|g(z_n)|}{R} = \frac{1}{2\pi} \log \frac{|b z_n|}{R}.$$

In particular, $|z_n g'(z_n)|/|g(z_n)| \to \infty$ as $n \to \infty$. On the other hand, $|z_n g'(z_n)|/|g(z_n)| = |(a-b)/b|$, a contradiction. By the assumption, $f(z)$ cannot be a polynomial. Thus f is a rational function. We assume $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + q(z)/p(z)$, where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$, q and p are two coprime polynomials with $\deg q < \deg p$, and n is a positive integer. Then, by Lemma 4,

$$f(z) = b(z-d) + c + \frac{A}{m(z-d)^m}, \qquad f'(z) = b - \frac{A}{(z-d)^{m+1}},$$

where c, d, $A \neq 0$, a_0 are constants, m is a positive integer.

If $a = 0$, i.e., $f(z) = 0 \Leftrightarrow f'(z) = 0$, then the zeros of f are all multiple. So the set $\{z \in C\colon f(z) = 0\}$ has at most $(m+1)/2$ distinct elements, while the set $\{z \in C\colon f'(z) = 0\}$ has $m + 1$ distinct elements. This contradicts the assumption that $f(z) = 0 \Leftrightarrow f'(z) = 0$.

Thus $a \neq 0$ and it follows that the zeros of function $f(z)$ and $f'(z) - a$ are all simple. Hence the function of $f(z)/(f'(z) - a)$ is entire on complex plane and has only one zero d which is the pole of $f(z)$. So we can get $f(z)/(f'(z) - a) \equiv p(z - d)^n$ where p is a constant and n is a positive integer. Hence,

$$\begin{aligned} &mb(z-d)^{m+2} + cm(z-d)^{m+1} + A(z-d) \\ &\quad \equiv (b-a)p(z-d)^{m+n+1} - p(z-d)^n A. \end{aligned}$$

So we obtain $n = 1$, $c = 0$, and $a = (m + 1)b$. The lemma is proved. □

Lemma 7. *Let* $P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \cdots + a_1(z)\omega$, $d(z)$, $a_i(z)$ $(i = 1, 2, \ldots, q-1)$ *analytic in* $\{z\colon |z| \leqslant 1\}$. *Then the set*

$$S = \left\{\omega \in C\colon\ P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \cdots + a_1(z)\omega = d(z),\ |z| \leqslant 1\right\}$$

is a bounded set.

Proof. We need only to prove that $|\omega|$ is bounded if $|\omega| \geqslant 1$ for $\omega \in S$. By the assumption, there exists a positive constant $M > 0$ such that $|a_i(z)| \leqslant M$ and $|d(z)| \leqslant M$ for $|z| \leqslant 1$. Further, by the assumption that $P(\omega) = d(z)$, we have

$$|\omega| = \left| a_{q-1}(z) + \cdots + a_1(z)\frac{1}{\omega^{q-2}} - d(z)\frac{1}{\omega^{q-1}} \right|.$$

So we can get $|\omega| \leqslant Mq + 1$ for each $\omega \in S$. The proof is completed. □

Lemma 8 [9]. *Let f be a meromorphic function and k be a positive integer,* $f^{(k)}(z) \not\equiv 0$. *Then for every* $\varepsilon > 0$

$$\overline{N}(r, f) \leqslant \frac{1}{k} N\left(r, \frac{1}{f^{(k)}}\right) + \frac{1}{k} N(r, f) + \varepsilon T(r, f) + S(r, f).$$

3. Proof of theorems

Proof of Theorem 1. We may assume that $D = \Delta$, the unit disc, and $a(z)$, $a_i(z)$ $(i = 1, 2, \ldots, q-1)$ analytic in $\{z\colon |z| \leqslant 1\}$. So there exists a positive constant $M > 0$ such that $|a(z)| \leqslant M$ and $|a_i(z)| \leqslant M$ $(i = 1, 2, \ldots, q-1)$ for $z \in \Delta$. By the assumption and Lemma 7 we have $|f^{(k)}(z_0)| \leqslant Mq + 1$ when $f(z_0) = 0$. Suppose that $\mathcal{F}$ is not normal at origin. Then by Lemma 1, for $A = Mq + 1$, there exist a sequence of function $f_n \in \mathcal{F}$, a sequence of complex numbers $z_n \to 0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\xi) = \rho_n^{-k} f_n(z_n + \rho_n \xi) \to g(\xi)$$

converges locally uniformly to a nonconstant function g, which is meromorphic in C and of finite order. Moreover, $g^{\#}(\xi) \leqslant g^{\#}(0) = kA + 1$ for all $\xi \in C$. Since $g_n(\xi)$ has only zeros of multiplicity at least k, by Hurwitz's theorem, the zeros of $g(\xi)$ are of multiplicity at least k.

Set

$$Q(\omega) = \omega^q + a_{q-1}(0)\omega^{q-1} + \cdots + a_1(0)\omega.$$

We claim:

(i) $g(\xi) = 0 \Leftrightarrow Q(g^{(k)}(\xi)) = a(0)$, and
(ii) $Q(g^{(k)}(\xi)) \neq b(0)$ on C.

In order to convenience in the following proof, we define

$$L(f) = P\big(f^{(k)}\big) + H\big(f, f', \ldots, f^{(k)}\big).$$

Suppose that $g(\xi_0) = 0$. Then by Hurwitz's theorem, there exist ξ_n, $\xi_n \to \xi_0$, such that $g_n(\xi_n) = \rho_n^{-k} f_n(z_n + \rho_n\xi_n) = 0$ (for n sufficiently large). Thus $L(f_n(z_n + \rho_n\xi_n)) = a(z_n + \rho_n\xi_n)$, i.e.,

$$\begin{aligned}
&L\big(f_n(z_n + \rho_n\xi_n)\big)\\
&\quad = P\big(f_n^{(k)}(z_n + \rho_n\xi_n)\big) + H\big(f_n(z_n + \rho_n\xi_n), \ldots, f_n^{(k)}(z_n + \rho_n\xi_n)\big)\\
&\quad = \big(f_n^{(k)}(z_n + \rho_n\xi_n)\big)^q + \sum_{i=1}^{q-1} a_i(z_n + \rho_n\xi_n)\big(f_n^{(k)}(z_n + \rho_n\xi_n)\big)^i\\
&\qquad + \sum_{i=1}^{m} b_i(z_n + \rho_n\xi_n) M_i\big(f_n(z_n + \rho_n\xi_n), \ldots, f_n^{(k)}(z_n + \rho_n\xi_n)\big)\\
&\quad = \big(g_n^{(k)}(\xi_n)\big)^q + a_{q-1}(z_n + \rho_n\xi_n)\big(g_n^{(k)}(\xi_n)\big)^{q-1} + \cdots + a_1(z_n + \rho_n\xi_n) g_n^{(k)}(\xi_n)\\
&\qquad + \sum_{i=1}^{m} b_i(z_n + \rho_n\xi_n)\rho_n{}^{(k+1)\gamma_{M_i} - \Gamma_{M_i}} M_i\big(g_n(\xi_n), \ldots, g_n^{(k)}(\xi_n)\big)\\
&\quad = a(z_n + \rho_n\xi_n).
\end{aligned}$$

From the assumption that $\gamma_H^* > 0$ we can know $\Gamma_{M_i}/\gamma_{M_i} < (k+1)$ for $i = 1, 2, \ldots, n$. Hence we can deduce that

$$\sum_{i=1}^{n} b_i(z_n + \rho_n\xi_n)\rho_n{}^{(k+1)\gamma_{M_i} - \Gamma_{M_i}} M_i\big(g_n(\xi_n), \ldots, g_n^{(k)}(\xi_n)\big)$$

converges uniformly to 0 on the neighborhood of the point ξ_0. In the limit as $n \to \infty$ we obtain $Q(g^{(k)}(\xi_0)) = a(0)$. This is $g(\xi) = 0 \Rightarrow Q(g^{(k)}(\xi)) = a(0)$.

Suppose now that $Q(g^{(k)}(\xi_0)) = a(0)$. We claim that $Q(g^{(k)}(\xi)) \not\equiv a(0)$. Indeed, otherwise $g^{(k)}(\xi) \equiv c_0$ where c_0 is a constant. By Lemma 7, we have $|c_0| \leqslant Mq + 1$. Since each zero of g has multiplicity at least k, g must have a single zero ξ_1 of multiplicity k, so that $g(\xi) = c_0(\xi - \xi_1)^k/k!$. A simple calculation then shows that

$$g^{\#}(0) \leqslant \begin{cases} k/2 & \text{if } |\xi_1| \geqslant 1,\\ |c_0| & \text{if } |\xi_1| \leqslant 1, \end{cases}$$

which contradict that $g^{\#}(0) = kA + 1$ and $A = Mq + 1$.

Since $Q(g^{(k)}(\xi_0)) = a(0)$ but $Q(g^{(k)}(\xi)) \not\equiv a(0)$ and

$$L\big(f_n(z_n + \rho_n\xi)\big) - a(z_n + \rho_n\xi) \quad \Rightarrow \quad Q\big(g^{(k)}(\xi)\big) - a(0)$$

on some neighborhood of the point ξ_0, there exist ξ_n, $\xi_n \to \xi_0$, such that $L(f_n(z_n + \rho_n\xi_n)) = a(z_n + \rho_n\xi_n)$. So $f_n(z_n + \rho_n\xi_n) = 0$. It is easy to deduce that $g(\xi_0) = 0$. This proves (i).

Next we prove (ii). Suppose $Q(g^{(k)}(\xi_0)) = b(0)$. Then $g(\xi_0) \neq \infty$. Further $Q(g^{(k)}(\xi)) \not\equiv b(0)$, otherwise that would imply $g^{(k)}(\xi) \equiv$ constant, which contradicts the conclusion (i). Thus, by Hurwitz's theorem, there exists ξ_n, $\xi_n \to \xi_0$, such that $L(f_n(z_n + \rho_n\xi_n)) = b(z_n + \rho_n\xi_n)$. Since $L(f_n(z)) = b(z) \Rightarrow f_n(z) = c(z)$ and $c(0) \neq 0$, we have $f_n(z_n + \rho_n\xi_n) = c(z_n + \rho_n\xi_n)$ and $g_n(\xi_n) = f_n(z_n + \rho_n\xi_n)/\rho_n^k = c(z_n + \rho_n\xi_n)/\rho_n^k \to \infty$, which contradicts $\lim_{n\to\infty} g_n(\xi_n) = g(\xi_0) \neq \infty$. This proves (ii).

Obviously, zero is not root of the polynomail $Q(\omega) - b(0)$. Since $b(0) \neq 0$ and the conclusion (ii), we have that there exist a nonzero constant b_1 such that $g^{(k)}(\xi) \neq b_1$. Suppose that g is a transcendental meromorphic function; by Lemma 5 and $g^{(k)}(\xi) \neq b_1$, we know g must have infinitely many zeros $z_1, z_2, \ldots, z_n, \ldots$. Define $h(\xi) = g^{(k-1)}(\xi) - b_1\xi$; then $h'(\xi) = g^{(k)}(\xi) - b_1$ and $h(\xi)$ is a transcendental meromorphic function of finite order. Hence, by Lemmas 2 and 3, there exists $R > 0$ such that

$$\frac{|z_n h'(z_n)|}{|h(z_n)|} \geqslant \frac{1}{2\pi} \log \frac{|h(z_n)|}{R} = \frac{1}{2\pi} \log \frac{|b_1 z_n|}{R}.$$

In particular, $|z_n h'(z_n)|/|h(z_n)| \to \infty$ as $n \to \infty$. On the other hand, by Lemma 7, we know $|z_n h'(z_n)|/|h(z_n)|$ be bounded when $n \to \infty$, a contradiction. So g is not a transcendental function. Suppose g be a polynomial. Since $g^{(k)}(\xi) \neq b_1$ and the zero of $g(\xi)$ are of multiplicity at least k, we can get g be a polynomial with the degree of k, which contradicts the conclusion (i).

Hence, $g(\xi)$ is a rational function. In the following, we consider two cases:

Case 1. $q \geqslant 2$. If the polynomial of $Q(\omega) - b(0)$ has only one root, then $Q(\omega) - b(0) = (\omega - \alpha)^q$, where $\alpha \neq 0$ is a constant. From the conclusion (ii) we can deduce that $(g^{(k)}(\xi) - \alpha)^q \neq 0$, i.e., $g^{(k)}(\xi) \neq \alpha$. By Lemma 4, we have

$$g(\xi) = \frac{\alpha\xi^k}{k!} + \cdots + a_0 + \frac{A}{(\xi - d)^m} \tag{1}$$

where $A \neq 0$, a_0 are constants, m is a positive integer.

So

$$\begin{aligned} Q\big(g^{(k)}(\xi)\big) - a(0) &= \big(g^{(k)}(\xi) - \alpha\big)^q - \big(a(0) - b(0)\big) \\ &= \left(\frac{A_1}{(\xi - d)^{m+k}}\right)^q - \big(a(0) - b(0)\big), \end{aligned}$$

where $A_1 \neq 0$ is a constant.

If $a(0) = b(0)$, then $Q(g^{(k)}(\xi)) - a(0) \neq 0$. According to the conclusion (i), we have $g(\xi) \neq 0$ which contradicts (1). So $a(0) \neq b(0)$. Since the zeros of g all have multiplicity at least k, the set $\{\xi \in C\colon\ g(\xi) = 0\}$ has at most $(m + k)/k$ distinct elements, while the set $\{\xi \in C\colon\ Q(g^{(k)}(\xi)) = a(0)\}$ has $(m + k)q$ distinct elements. This contradicts the conclusion (i).

If the polynomail $Q(\omega) - b(0)$ has at least two distinct roots, then there exist b_1, b_2, $b_1 \neq b_2$, $b_1 b_2 \neq 0$ such that $g^{(k)}(\xi) \neq b_i$ for $i = 1, 2$. By Lemma 4 and $g^{(k)}(\xi) \neq b_1$, we have $g^{(k)}(\xi) = b_1 + A/(\xi - d)^m$, which contradicts $g^{(k)}(\xi) \neq b_2$.

Case 2. $q = 1$. The conclusions (i) and (ii) can be written as (i) $g = 0 \Leftrightarrow g^{(k)} = a(0)$ and (ii) $g^{(k)} \neq b(0)$.

If $k \geqslant 2$, by Lemma 4 and $g^k \neq b(0)$ and $b(0) \neq 0$, we have

$$g(\xi) = \frac{b(0)\xi^k}{k!} + \cdots + a_0 + \frac{A}{(\xi - d)^m}, \qquad g^{(k)}(\xi) = b(0) + \frac{A_1}{(\xi - d)^{m+k}}, \tag{2}$$

where d, $A \neq 0$, $A_1 \neq 0$, a_0 are constants, m is a positive integer.

If $a(0) = b(0)$, then $g^{(k)}(\xi) - a(0) \neq 0$. According to the conclusion (i), we have $g(\xi) \neq 0$ which contradicts (2). So $a(0) \neq b(0)$. Since the zeros of g all have multiplicity at least k, the set $\{\xi \in C\colon\ g(\xi) = 0\}$ has at most $(m + k)/k$ distinct elements, while the set $\{\xi \in C\colon\ g^{(k)}(\xi) = a(0)\}$ has $(m + k)$ distinct elements. This contradicts the conclusion (i).

If $k = 1$, by Lemma 6, we have $a(0) = (m + 1)b(0)$, which contradicts the assumption. This completes the proof of theorem. □

Proof of Theorem 2. When $k = 1$, this is the Theorem B [2]. So we assume that $k \geqslant 2$.

We may assume that $D = \Delta$, the unit disc. Suppose that $\mathcal{F}$ is not normal in D; without loss of generality, we assume that $\mathcal{F}$ is not normal at $z_0 = 0$. Then by Lemma 1, there exist, for $\alpha = 0$,

(a) a number r, $0 < r < 1$,
(b) points z_n, $|z_n| < r < 1$, $z_n \to 0$,
(c) functions $f_n \in \mathcal{F}$, and
(d) positive numbers $\rho_n \to 0^+$,

such that

$$g_n(\xi) = \big(f_n(z_n + \rho_n \xi) - c\big) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where $g(\xi)$ is a meromorphic function on C which satisfies $g^{(k)}(\xi) \not\equiv 0$.

We claim that $g^{(k)} \neq 0$, $g \neq a - c$, and $g \neq b - c$.

Indeed, suppose $g^{(k)}(\xi_0) = 0$. Since $g_n^{(k)}(\xi) - \rho_n^k a \to g^{(k)}(\xi)$, and $g^{(k)} \not\equiv 0$, there exist ξ_n, $\xi_n \to \xi_0$, such that

$$g_n^{(k)}(\xi_n) - \rho_n^k a = 0, \quad \text{i.e.,} \quad f_n^{(k)}(z_n + \rho_n \xi_n) = a.$$

Since f and $f^{(k)}$ share a, we have

$$g_n(\xi_n) = f_n(z_n + \rho_n \xi_n) - c = a - c.$$

It follows that

$$g(\xi_0) = \lim_{n \to \infty} g_n(\xi_n) = a - c. \tag{3}$$

Similarly, $g_n^{(k)}(\xi) - \rho_n^k b \to g^{(k)}(\xi)$. Thus there exist $\xi_n^*, \xi_n^* \to \xi_0$, such that

$$g_n^{(k)}\big(\xi_n^*\big) - \rho_n^k b = 0, \quad \text{i.e.,} \quad f_n^{(k)}\big(z_n + \rho_n \xi_n^*\big) = b.$$

Since f and $f^{(k)}$ share b, we have

$$g_n(\xi_n^*) = f_n(z_n + \rho_n \xi_n^*) - c = b - c.$$

It follows that

$$g(\xi_0) = \lim_{n\to\infty} g_n(\xi_n^*) = b - c. \tag{4}$$

According to (3) and (4), we deduce that $a = b$, contradicting the hypothesis.

Now, we prove $g \neq a - c$ and $g \neq b - c$. Suppose $g(\xi_0) = a - c$, by Hurwitz's theorem, there exist ξ_n^{**}, $\xi_n^{**} \to \xi_0$ such that

$$f_n(z_n + \rho_n \xi_n^{**}) = a.$$

Since f and $f^{(k)}$ share a, we have

$$\rho_n^k f_n^{(k)}(z_n + \rho_n \xi_n^{**}) - \rho_n^k a = 0.$$

Let $n \to \infty$; we have $g^{(k)}(\xi_0) = 0$ which contradict $g^{(k)} \neq 0$.

By the same method, we can get $g \neq b - c$.

Now, by the second fundamental theorem of Nevanlinna, we have

$$T(r, g) \leqslant \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g - (a - c)}\right) + \overline{N}\left(r, \frac{1}{g - (b - c)}\right) + S(r, g).$$

By Lemma 8, taking $\varepsilon = 1/4$ and attending $k \geqslant 2$, we have

$$T(r, g) \leqslant O(1)S(r, g).$$

So g is a rational function. According to $g \neq a - c$ and $g \neq b - c$ we can deduce g is a constant, a contradiction. This completes the proof. □

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